**Example of Using Direct Determined Probabilistic Solution in SBRA Method**

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**Introduction**

SBRA method (Simulation Based Reliability Assessment [3]) is developed from the second half of eightieth. It was used in series of problems published in many articles and textbooks (e.g. [6]). In application of this method are input random variables (load, geometric and material properties, imperfections etc.) expressed by histograms. A transformation model derived from the analyses was employed to determine the response of the structure to the load. This transformation model and Monte Carlo were used to calculate the probability of defects in the support structures. The final reliability resulted from the comparison of such defect and designed value of the defect probability.

This paper is based on the example of the alternative using direct determined probabilistic calculation. No Monte Carlo simulation, used in SBRA, has been employed, but the same transformation model and, generally, same input data have been used. Example of frame was taken from [4], though it was developed in [5].

**Principles of Direct Determined Probabilistic Solution**

The procedure of the calculation above was published first in [1] and more developed in [2]. Same input data as for the SBRA method are being used: this means, the input data are random quantities expressed by histograms. When applying Monte Carlo, one input quantity is selected (generated) on a random basis from each set of input random quantities for the function concerned (from each histogram), the aim being to find the value of that function (In SBRA method input random variables are approximated by piece wise uniform -PWU). The number of selections carried out from each input quantity, and performed for each input quantities, equals to the number of numeric simulations. In case of the direct determined probabilistic solution, the input quantities are also expressed by histograms (with discrete distribution), however they are neither selected on a random basis nor generated. Such quantities enter the calculation on the determined basis, exactly pursuant to the set algorithm. The quality of the result can be same as that reached when using Monte Carlo: e.g. the histogram of that function. If the input histograms, functions, and intervals for the input quantities are the same, the result will always be the same for the direct determined probabilistic solution. With Monte Carlo, the result is always somewhat different even if the input quantities, functions, and the number of simulations are same because the generated input quantities are not same, they are selected on a random basis, and the number of simulations is almost always finite in practice.
The direct determined probabilistic method is based on key definitions and probability theory methods some of which will be mentioned [7].

A random event is an event that may or may not occur under certain conditions.

Probability is a quantitative expression of such random event.

If, under certain conditions, one of \( n \) mutually exclusive events should occur, and there is no reason to anticipate than any of such events is more likely to occur than the others, such events are said to have the same probability.

\[
/1/ \quad p = \frac{1}{n}
\]

If any random event \( A \) is consequence of any of \( m \) events, \( n \) being the number of possible (mutually exclusive and equally probable) events then the probability of the event \( A \) is expressed as follows:

\[
/2/ \quad p = \frac{m}{n}
\]

The probability of the present occurrence of several independent events equals to the product of probabilities of such events, while the probability of occurrence of the same event out of several mutually exclusive events equals to the sum of probabilities of such events.

The theses above represent the basis of the direct determined probabilistic calculation.

Let the histogram \( B \) be any function \( f \) of histograms \( A_j \), with \( j \) ranging from 1 to \( n \). This means:

\[
/3/ \quad B = f(A_1, A_2, A_3, \ldots, A_j, \ldots A_n),
\]

where \( n \) is number of histograms \( A_j \).

Each histogram \( A_j \) includes \( i_j \) intervals, and each interval is limited by the lower limit \( a_{j,i} \) and upper limit \( a_{j,i+1} \). Consequently, the interval \( i_j = 1 \) includes following values:

\[
/4/ \quad a_{j,1} \leq a_j \leq a_{j,2},
\]

and

\[
/5/ \quad a_{j,2} = a_{j,1} + \Delta a_j,
\]

and

\[
/6/ \quad \Delta a_j = \frac{a_{j,max} - a_{j,min}}{N_j} \quad \text{and} \quad N_j \text{ is number of intervals in histogram } A_j.
\]

In the interval \( i_j \) the following holds good generally:

\[
/7/ \quad a_{j,i} \leq a_j \leq a_{j,i+1}
\]
Let’s refer to values \(a_j\) within this interval as \(a_j^{(ij)}\). Analogous expressions apply to the histogram \(B\). If there are \(N\) intervals, histogram values in the \(i^{th}\) interval range from \(b_i\) to \(b_{i+1}\), (hereinafter referred to as \(b^{(i)}\)) resulting from the following function:

\[
/8/ \quad b^{(i)} = f(a_1^{(i1)}, a_2^{(i2)}, \ldots, a_j^{(ij)}, \ldots, a_n^{(in)})
\]

for the certain combination of arguments \(a_1^{(i1)}, a_2^{(i2)}, \ldots, a_j^{(ij)}, \ldots, a_n^{(in)}\). The same value - \(b^{(i)}\) - can be however achieved with other values (or at least with some other values) \(a_j^{(ij)}\). If the possible combination of values \(a_j^{(ij)}\) is referred to as \(l\), the following holds good:

\[
/9/ \quad b^{(i)} = f(a_1^{(i1)}, a_2^{(i2)}, \ldots, a_j^{(ij)}, \ldots, a_n^{(in)})_l
\]

The probability \(p_{b^{(i)}}\) of occurrence \(b^{(i)}\) is the product of probabilities \(p_{aj^{(ij)}}\) of occurrence of values independent events \(a_j^{(ij)}\). This means:

\[
/10/ \quad p_{b^{(i)}} = (p_{aj^{(i1)}} \cdot p_{aj^{(i2)}} \cdot p_{aj^{(i3)}} \cdot \ldots \cdot p_{aj^{(ij)}} \cdot \ldots \cdot p_{aj^{(in)}})
\]

The probability of occurrence of all possible combinations \((a_1^{(i1)}, a_2^{(i2)}, \ldots, a_j^{(ij)}, \ldots, a_n^{(in)})_l\) of the function \(f\) the result of which is \(b^{(i)}\) is as follows:

\[
/11/ \quad p_{b^{(i)}} = \sum_{l=1}^{j} p_{pl}^{(i)}
\]

The number of intervals \(N_j\) within each histogram \(A_j\) as well as the number of intervals \(i\) in the histogram \(B\) can be different. It is however just the number of intervals that has the decisive and major impact on the accuracy of the calculation, number of necessary mathematical operations and time needed for calculation.

Principles of operations carried out with the two histograms referred to as \(A_j\) and \(A_i\) is given in the schematic sketch, fig 1. It is evident that the values from within the interval from \(b_i\) to \(b_{i+1}\) are the result of three different combinations of the values \(a_1, a_{ij}, \ldots, a_n\) this means the result of three different values from within the intervals \(a_1^{(i1)}\) to \(a_{ij}^{(ij)}\) and \(a_{ij}^{(ij)}\) to \(a_{ij}^{(ij)}\). Therefore, these combinations are independent of each other, and probabilities of the values from within the interval \(b_i\) to \(b_{i+1}\) are as follows: \(p_{b^{(i)}} = p_{b1^{(i)}} + p_{b2^{(i)}} + p_{b3^{(i)}}\).

The probabilistic calculations of challenging tasks especially need much time and resources. What is of key importance is the calculation of random quantities used as inputs for the tasks and the calculation of selected intervals for each variable. For the task concerned, the number of variables is clearly given. The selection of the number of intervals to be used for each variable is arbitrary, to a certain extent. Generally, the number of intervals should be such that the time of the calculation could be feasible, and an increase in the number of intervals should not have considerable impacts on the result. It is assumed that the more intervals are used, the more accurate the results are. This is however true to a certain extent only.

If the number of the histograms \(A_j\) equals to \(n\), and the number of intervals in the histogram \(A_j\) is \(N_j\), then the number of intervals in the histogram \(B\) will be generally:

\[
/12/ \quad N \leq N_1 \cdot N_2 \cdot N_3 \cdot \ldots \cdot N_j \cdot \ldots \cdot N_n
\]

The number of mathematical calculations is direct proportional to the following product:

\[
/13/ \quad P = N_1 \cdot N_2 \cdot N_3 \cdot \ldots \cdot N_j \cdot \ldots \cdot N_n
\]

and for
It is obvious that there is no generally reason to select $N > P$. For practical reasons and calculation accuracy, it is however recommended to select: $N \ll P$. For instance, if $n = 3$ and $N_j = 100$, then the resulting $P = (100)^3 = 10^6$ pursuant to /14/. The number of intervals in the histogram $B$ can be even $N = 10^6$. Generally it is enough to choose the number $N$ where the number of intervals in the histogram $B$ will be lower by an order.

For $N_j = 1000$ and with the same $n = 3$, the resulting $P = (10)^9$. It is evident that the number of calculations performed for $N_j$ goes up, and the time of calculations becomes, in turn, longer too. If $n = 20$ and $N_j = 256$, then $P = (256)^{20} = 1,4615 \times 10^{48}$. This is the number of operations which is not feasible in practice even if state-of-the-art computer systems are used.

A logical question should have been asked: can the probabilistic calculation be used at all? In what cases the probabilistic calculation can be used? Are the possibilities of reducing the number of mathematical operations for the task concerned, maintaining at the same time the correctness of the final solution?

It is evident that, for instance, if two histograms are added up ($B = A_1 + A_2$), the number of operation is $P = N_1 \cdot N_2$. For $N_1 = N_2 = 128$, the resulting $P = 1282 = 16384$, and for $N_1 = N_2 = 256$, the resulting $P = 256^2 = 65536$.

Any calculation of such kind is very fast even if ordinary computer systems are used. The situation is the same if $B$ is given by a different function of the histograms $A_1$, $A_2$. If $B$ is the function of a rather big number of the histograms $A_j$, attention must be paid to if the number of the mathematical operations can be reduced in order to have the calculation feasible and results sufficiently accurate.
Recently, following possibilities for reducing the required number of operations have appeared:

1. Reduction of the number of intervals for the random input quantities in individual histograms, maintaining at the same time the whole scope for each random input quantity.

2. Reduction of the number of intervals of random input quantities for each histogram included into the calculation, maintaining at the same time the total quantity of intervals. For purposes of the calculation, used are, only or mostly only, the intervals which have impacts on the value under investigation, when investigating for instance the structure defect.

3. Putting into groups of such random input quantities, which could influence the calculation jointly, if a joint histogram can be prepared for such quantities in advance. (For the mathematical operation with the random input quantities, the commutative and associative laws are used. No distributive law is permissible for that purpose). The pre-processing of the quantities must be correct in order to ensure that the result of the joint histogram used for the calculation could be the same as if individual histograms were employed for the calculation.

4. Statistics-dependant or function-dependant random input quantities are not included into the calculation separately, but, if possible, a joint histogram or histograms are used (considering the strength and deformation properties of the material and/or cross-section characteristics).

5. Combination of the methods above.

The procedure below makes use and applies all mentioned methods except for 2) which has not been finalised in the programme yet.
Assessment of Reliability of Steel Support Structure using SBRA by Direct Determined Probabilistic Calculation

The structure under assessment has been taken from [4]. The static chart for this structure is given in Fig. 2. The structure is a frame with one static indeterminacy. For the relocation $\delta$ the following formula can be used:

$$\delta = \frac{W + EQ + \sum_{i=1}^{4} \frac{a_i F_i}{l_i}}{(1 + \frac{F_2}{F_1} K_1 l_1) - \sum_{i=3}^{4} \frac{F_i}{l_i}}$$

The formula can be modified as follows:

$$\delta = \frac{W + EQ + \frac{a_1 F_1}{l_1} + \frac{a_2 F_2}{l_2} + \frac{a_3 F_3}{l_3} + \frac{a_4 F_4}{l_4}}{\frac{F_1}{l_1 K_1} + \frac{F_2}{l_2 K_2} - \frac{F_3}{l_3} - \frac{F_4}{l_4}}$$

In the formulae /15/ and /16/:

$$K_1 = \frac{\frac{F_1}{l_1}}{\sqrt{\frac{EI}{l_1}}} - 1$$

and

$$K_2 = \frac{\frac{F_2}{l_2}}{\sqrt{\frac{EI}{l_2}}} - 1$$

The bending moments in fixing are:

$$M_1 = \frac{\delta (1 + K_1)}{K_1} F$$

and

$$M_2 = \frac{\delta (1 + K_2)}{K_2} F_2$$

The stress in the outer fixity fibres is:

$$\sigma_1 = \left| \frac{M_1}{W_1} \right| + \left| \frac{F_1}{A_1} \right| = F_1 \left( \left| \frac{\delta (1 + K_1)}{K_1 W_1} \right| + \frac{1}{A_1} \right)$$
\[ \sigma_2 = \frac{M_2}{W_2} + \frac{F_2}{A_2} = F_2 \left( \frac{\delta^2 (1 + K_2)}{K_2 W_2} + \frac{1}{A_2} \right) \]

For each column, the reliability function has been analysed for the load limits

\[ SF = R - Q \]

where \( R \) stands for the limit of material properties, and \( Q \) stands for the extreme stress in the column fixity pursuant to \( /21/ \) a \( /22/ \).

The serviceability of the structure is related to the permissible horizontal shift of upper cross bars being \( \delta_{tol} = 35 \text{ mm} \). The serviceability function is:

\[ SF = \delta_{tol} - \delta \]

where \( \delta \) stands for the actual horizontal shift of the frame pursuant to \( /15/ \) and \( /16/ \).

**Input:**

Following abbreviations are used for the formula above:

\( W \) wind load, variables

\( EQ \) earthquake load, variables

\( F_1, F_2, F_3, F_4 \) axial forces in columns - they consist of the permanent variable load, long-term random load, and short-term random load

\( a_1, a_2, a_3, a_4 \) imperfections in columns, variables

\( l_1, l_2, l_3, l_4 \) column lengths, permanent value, no variables

\( J_1, J_2 \) inertia moments for columns 1 and 2, variables

\( W_1, W_2 \) cross-section moduli for columns 1 and 2, variables

\( A_1, A_2 \) cross-section areas for columns 1 and 2, variables

\( E \) modulus of elasticity, no variable, \( E = 2.1 \times 10^8 \text{ kPa} \)

\( R \) yield strength, variable.

**Variables:**

25 variables are used in the calculation pursuant to the formulae \( /15/ \) to \( /24/ \): Namely:

- load values \( F_1, F_2, F_3, F_4, W \) and \( EQ \). Each value \( F_i \) is expressed by three independent histograms (for the permanent load, long-term random load, and short-term random load). The load values result from 14 histograms, each of them consisting of 256 intervals.

- imperfection values \( a_1, a_2, a_3, a_4 \). These 4 values can be positive or negative. The number of the intervals is the same as for the histograms below: 256.
- inertia moments $J_1, J_2$, cross-section moduli $W_1, W_2$, and cross-section area $A_1, A_2$ for the first and second columns. The total is 6 values. In article [1], these values are regarded as absolutely independent for the purposes of the calculation.

- yield strength of the material.

If all possible combinations were taken into account and the corresponding number of mathematical operations were carried out, the resulting $P = 256^{25} = 1,606938.10^{60}$. It is beyond doubt that this number of mathematical operations is not currently feasible in real time. The number of operations can be however reduced considerably. Following possibilities are available:

1. The wind load $W$ and earthquake load $EQ$ occur in the formula /15/ are random and independent variables. They can be added up and a joint histogram can be created for the both loads.

2. Each load from among $F_1, F_2, F_3, F_4$ is expressed through three independent histograms. For each force $F_i$ an only one histogram can be used which is the sum of three histograms consisting of the above mentioned independent loads.

3. The cross-section characteristics are regarded as the quantities with the mutual functional dependency. Therefore, only one histogram can be used to describe them.

4. For each histogram, 8 intervals have been chosen, except for the earthquake/wind load histogram with 9 intervals and alternatively with 52 and 252 intervals.

When calculating the shift pursuant to /16/, there will be only 11 histograms for independent input variables. When calculating the stress pursuant to /21/ and /22/, there will be also 11 histograms only provided that the cross-section characteristics are considered to be the function-dependant quantities which can be described with one histogram only. When calculating the deformation $\delta$ and the stress in the columns 1 and 2, the number of the operations will be $P = 810.9 = 9663\ 676\ 416 = 9,664.10^9$, for 9 intervals by point 4.

Even this number is rather high, and if less intervals are used, the final results can be influenced to a certain extent. In order to provide complete information it should be pointed out that the histograms created by the calculation of the forces pursuant to the points 1) and a) had originally 256 intervals, and the number of the intervals has been reduced step by step. The resulting histograms for the stress in the columns 1 and 2 have been compared, pursuant to /23/, with the histogram prepared for the yield strength of the material, pursuant to [3] (that histogram consisted of 102 intervals). If the same critical approach could be taken in respect of the reduction of the intervals from 256 down to 8 or 9, it is believed that other methods resulting in the reduction of the mathematical operations would be absolutely correct.

The assumptions above have been partly verified by calculations where the number of intervals has been changed which characterise the wind and earthquake load ($W + EQ$). This histogram is, beyond doubt, considerably affected by the horizontal shift. First, the number of intervals was 9 (see the number of operations above), then it was increased up to 52, and finally there were 256 intervals. For the second calculation, the number of operation is:

$$P = 8^{10}.52 = 55\ 834\ 574\ 848 = 5,583.10^{10}, \text{ and for the third calculation:}$$

$$P = 8^{10}.256 = 274\ 877906\ 948 = 2,749.10^{11}. $$
The increase in the number of intervals has affected the results - see below. Such impacts are also expected, if the number of intervals is increased for other input quantities.

**Assessment of the Frame:**

The reliability assessment is based on the analysis of the reliability functions /23/ and /24/. The reliability criterion is expressed by $P_f < P_d$, where $P_f$ is the probability of failure and $P_d$ is the target probability.

To assess the reliability of the frame based on the principles above, the application software developed under Borland Delphi has been used.

Fig.3 gives the introductory form where characteristic values and histograms need to be entered for individual variables. At the bottom of Fig. 3 and in Fig. 4 too, there is the resulting histogram for the horizontal deformation $\delta$. The limit values for that histogram are given in Table 1.

![Fig.3. Programme Output: Introductory Screen and Horizontal Deformation $\delta$ Histogram. Wind/earthquake load histograms have been used for this solution with a) 9 intervals, b) 52 intervals, and c) 256 intervals.](image)

The assessment for the load limit based on the reliability function analysis pursuant to /23/ has been carried out in the fixity cross-section in the columns 1 and 2. The resulting histograms for the both reliability functions are given in the figures 5 and 6.

![Fig.4. Programme Output: Horizontal Deformation $\delta$ Histogram. Wind/earthquake load histograms have been used for this solution with a) 9 intervals, b) 52 intervals, and c) 256 intervals.](image)
The reliability related to the limit serviceability at the permissible horizontal shift of the upper cross bars $\delta_{tol}$ has been assessed by analysing the reliability function pursuant to /24/. The final histogram for this function is given in Fig.7.
Table 1 Reliability Assessment of Steel Support Structure by Direct Determined Probabilistic Calculation.

<table>
<thead>
<tr>
<th>number of intervals in histograms for wind / earthquake load</th>
<th>9</th>
<th>52</th>
<th>256</th>
</tr>
</thead>
<tbody>
<tr>
<td>number of simulations</td>
<td>9,66.10⁹</td>
<td>55,83.10⁹</td>
<td>274,88.10⁹</td>
</tr>
<tr>
<td>δ - horizontal deformation [m]</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>minimum</td>
<td>-0.20980</td>
<td>-0.20211</td>
<td>-0.20961</td>
</tr>
<tr>
<td>maximum</td>
<td>0.21030</td>
<td>0.20974</td>
<td>0.20974</td>
</tr>
<tr>
<td>load limits</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>column 1</td>
<td>$P_f = 0.000002469322 &lt; P_d = 0.00000080$</td>
<td>$P_f = 0.000002448392 &lt; P_d = 0.00000080$</td>
<td>$P_f = 0.000016701146 &lt; P_d = 0.00000700$</td>
</tr>
<tr>
<td>level of reliability</td>
<td>increased</td>
<td>increased</td>
<td>normal</td>
</tr>
<tr>
<td>column 2</td>
<td>$P_f &lt;&lt; 1.10^{-12}$</td>
<td>$P_f &lt;&lt; 1.10^{-12}$</td>
<td>$P_f &lt;&lt; 1.10^{-12}$</td>
</tr>
<tr>
<td>level of reliability</td>
<td>increased</td>
<td>increased</td>
<td>increased</td>
</tr>
<tr>
<td>serviceability limits</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>cross bars</td>
<td>$P_f = 0.046753096590 &lt; P_d = 0.070$</td>
<td>$P_f = 0.029341188347 &lt; P_d = 0.070$</td>
<td>$P_f = 0.024361125478 &lt; P_d = 0.070$</td>
</tr>
<tr>
<td>level of reliability</td>
<td>normal</td>
<td>normal</td>
<td>normal</td>
</tr>
</tbody>
</table>

Table 1 lists the resulting probabilities of the defects in the fixity of the columns 1 and 2 pursuant to the load limits which are caused by the overcome of the stress in the critical cross-section. This table also lists the resulting probability of defect arisen if the permissible horizontal shift of the upper cross bars is exceeded. Pursuant to the set criteria, the structure will be compliant for the both limit conditions with the normal level of reliability.

The results are not completely in line with the result published in [4] where, for same input values (the same geometrical dimensions of the structure, load, and material properties), the probability of failure for the assessment pursuant to the limit load of the column 1 is $P_f = 7.3 \times 10^{-5}$ (the decreased reliability) and of the column 2 is $P_f < 1.10^{-6}$ (the increased reliability), and the resulting limit serviceability is $P_f = 11.8 \times 10^{-2}$ (the decreased reliability).

There are definitely several reasons for the differences in the results. Certain role can be played by the entry of the cross-section characteristics of the columns which have been entered as mutually independent quantities in [4]. Because the calculation needs to be carried out in real time, the accuracy of the submitted solution is affected by the so-far limited number of intervals used for the input quantity histograms. It is believed however that after the direct determined probabilistic calculation is developed more in detail, the solution will be correct and sufficiently accurate even for the challenging example above.
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References


